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A Note on the Split Common Fixed Point Equality Problems in Hilbert Spaces

Kılıçman, A. *[1,2] and Mohammed, L. B.[1]

[1]*Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, Malaysia*
[2]*Institute for Mathematical Research, Universiti Putra Malaysia, Malaysia*

E-mails: akilic@upm.edu.my
* *Corresponding author*

ABSTRACT

We study the split common fixed point equality problems (SCFPEP). Furthermore, we formulate and analyse the algorithms for solving this SCFPEP for the finite family of quasi-nonexpansive operators in Hilbert spaces and shows how it unifies and generalizes previously discussed problems. In the end, we give numerical example that illustrates our theoretical results.

Keywords: Iterative Algorithm, quasi-nonexpansive, split feasibility problem, weak convergence.

1. Introduction

Let H_1, H_2 and H_3 be Hilbert spaces, $A : H_1 \to H_2$ and $B : H_2 \to H_3$ be bounded linear operators with their adjoint A^* and B^*, respectively. The mapping $S : H_1 \to H_1$ is called quasi-nonexpansive if

$$\|Sx - z\| \leq \|x - z\|, \forall x \in H_1 \text{ and } z \in Fix(S),$$

where $Fix(S)$ is the fixed point set of S, that is $Fix(S) = \{z \in H_1 : Sz = z\}$.

The split common fixed point problem (SCFPP) was introduced by Censor and Segal (2009). Since the inception of SCFPP in 2009, the problem has observed an explosive growth and found its application in the field of intensity-modulated radiation therapy (IMRT), for more details, see Censor et al. (2006).

The SCFPP is obtained as:

$$\text{Find } \; x^* \in \bigcap_{r=1}^{N} Fix(U_r) \; \text{ and } \; Ax^* \in \bigcap_{s=1}^{M} Fix(T_s), \tag{1}$$

where $U_r : H_1 \to H_1, r = 1, 2, 3, ..., N$, and $T_s : H_2 \to H_2, s = 1, 2, 3, ..., M$, are nonlinear mappings with $Fix(U_r) \neq \emptyset$ and $Fix(T_s) \neq \emptyset$, respectively.

We now consider the following problem "Split Common Fixed Point Equality Problems (in short, SCFPEP)", this is fomulated as:

$$\text{Find } x^* \in \bigcap_{r=1}^{N} Fix(U_r) \text{ and } y^* \in \bigcap_{s=1}^{M} Fix(T_s) \text{ such that } Ax^* = By^*, \tag{2}$$

where $U_r : H_1 \to H_1, r = 1, 2, 3, ...N$, and $T_s : H_2 \to H_2, s = 1, 2, 3, ...M$, are quasi-nonexpansive mappings with $Fix(U_r) \neq \emptyset$ and $Fix(T_s) \neq \emptyset$, respectively.

Note that, Problem (2) reduces to Problem (1) as $H_2 = H_3$ and $B = I$ (the identity operator on H_2). In the light of this, it is worth to mention here that the SCFPEP generalizes the SCFPP. Therefore, the results and conclusions that are true for the SCFPEP continue to hold for the SCFPP, and it shows the significance and the range of applicability of SCFPEP.

The notion of the split equality fixed point problems was introduced by Moudafi (2014) as a generalization of the split feasibility problem (SFP). The split equality fixed point problems (in short, SEFPP) is obtained as finding a vector

$$x^* \in C \text{ and } y^* \in Q \text{ such that } Ax^* = By^*. \tag{3}$$

Trivially, Problem (3) reduces to the following problem as $H_2 = H_3$ and $B = I$ (the identity operator on H_2), $C := Fix(T)$ and $Q := Fix(U)$:

$$x^* \in Fix(T) \text{ such that } Ax^* \in Fix(U). \tag{4}$$

This is called split feasibility problem (SFP).

Remark 1.1. *In Problem (1), if $r = s = 1$, we immediately obtain Problem (3). We already mentioned that SCFPEP reduces to the SCFPP. In the light of this, it is worth to mention that SCFPEP generalizes the SCFPP, SEFPP and SFP. Studying the SCFPEP will help in studying these problems.*

To approximate the solution of the SCFPP, Censor and Segal (2009) gave the weak convergence result which involving the class of cutter operators. This operator was introduced by Bauschke and Combettes (2001), see also Combettes (2001) and references therein. Recently, Moudafi (2011) proposed the following algorithm for solving the SCFPP for the class of demicontractive mappings and obtained the weak convergence results.

$$\begin{cases} u_n = x_n + \lambda A^*(T-I)Ax_n \\ x_{n+1} = (1-t_n)u_n + t_n U(u_n), \forall n \in \mathbb{N} \end{cases} \tag{5}$$

where $\lambda \in (0, (1-\mu)/\gamma)$ with $\gamma = A^*A$, $t_n \in (0,1)$ and $x_0 \in H_1$ was chosen arbitrarily.

Setting $t_n = 0$ in Algorithm (5), we immediately obtained

$$x_{n+1} = U\Big(x_n + \gamma A^*\big(T-I\big)Ax_n\Big), \forall n \geq 0. \tag{6}$$

This is exactly the original algorithm proposed by Censor and Segal (2009). Moudafi (2010) proposed another iterative method for solving the SCFPP and obtained the weak convergence result of the proposed algorithm. Related work can be found in Kilicman and Mohammed (2016), Mohammed and Kılıçman (2015) and references therein.

Very recently, Moudafi and Al-Shemas (2013) proposed the following simultaneous algorithm which generates the sequences $\{(x_n, y_n)\}$ by

$$\begin{cases} x_{n+1} = U(x_n - \lambda_n A^*(Ax_n - By_n), \\ \\ y_{n+1} = T(y_n + \lambda_n B^*(Ax_n - By_n), \forall n \geq 1, \end{cases} \tag{7}$$

After some suitable assumptions imposed on the parameters and operators involved, Moudafi and Al-Shemas (2013) proved that the sequences $\{(x_n, y_n)\}$ defined by Algorithm (7) converges weakly to the solution of SEFPP (3) whenever this problem exist.

Byrne and Moudafi (2012) investigated the following algorithm by using the Landweber's projection method:

$$\begin{cases} x_{n+1} = P_C(x_n - \lambda_n A^*(Ax_n - By_n), \\ y_{n+1} = P_Q(y_n + \lambda_n B^*(Ax_n - By_n), \forall n \geq 1, \end{cases} \tag{8}$$

where P is a metric projection. Trivially, Algorithm (8) is a particular case of Algorithm (7) as $U = P_C$ and $T = P_Q$.

Based on the work of Moudafi and Al-Shemas (2013); Ma et al. (2013) considered the following algorithm:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n U(x_n - \lambda_n A^*(Ax_n - By_n), \\ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n T(y_n + \lambda_n B^*(Ax_n - By_n), \forall n \geq 1, \end{cases} \tag{9}$$

where U, T, A, A^*, B, B^*, λ_n, L_1 and L_2 as in Algorithm (7), and $\alpha_n \subset [\alpha, 1]$ for $\alpha > 0$. By imposing some appropriate conditions on the parameters and operators involved, they obtained the weak convergence results for the solution of SEFPP (3).

Inspired by the work of Byrne and Moudafi (2012), Ma et al. (2013), Moudafi and Al-Shemas (2013), we will further consider an algorithm for solving the SCFPEP (2) for the finite family of quasi-nonexpansive mappings in Hilbert spaces, in the end, we give the convergence results of the proposed algorithm.

2. Preliminaries

This section gives some preliminaries results which were used in proving our main result.

Definition 2.1. *A mapping $T : H_1 \to H_1$ is said to be;*

(i) Demiclosed at zero, if for each $\{z_n\} \subset H_1$ with $z_n \rightharpoonup z$ and $Tz_n \to 0$, implies that $Tz = 0$.

(ii) Semi-compact if for any bounded sequence $\{z_n\} \subset H_1$ *with* $(I-T)z_n \to 0$, *and* $\exists$ $\{z_{n_k}\}$ *of* $\{z_n\}$ *such that* $z_{n_k} \to z \in H_1$.

Lemma 2.1. (Opial (1967)). *Let* $\{x_n\} \subset H_1$, *and* C *be a nonempty subset of* H_1 *such that the following conditions are satisfied:*

(i) For each $z \in C$, $\lim_{n\to\infty} \|z_n - z\|$ *exist,*

(ii) Any weak-cluster point of the sequence $\{z_n\}$ *belongs to* C.

Then $\exists$ $y \in C$ *such that* $z_n \rightharpoonup y$.

Lemma 2.2. (Li and He (2015)). *Let* $T_k : H_1 \to H_1$, *for k=1,2,3,...,M be M-quasi-nonexpansive mappings and defined* $U = \sum_{k=1}^{M} \delta_k U_{\gamma_k}$, *where* $U_{\gamma_k} = (1-\gamma_k)I + \gamma_k T_k$ *and* $\delta_k \in (0,1)$ *such that* $\sum_{k=1}^{M} \delta_k = 1$. *Then*

(i) U is a quasi-nonexpansive operator,

(ii) $Fix(U) = \bigcap_{k=1}^{M} Fix(U_{\gamma_k}) = \bigcap_{k=1}^{M} Fix(T_k)$,

(iii) in addition, if $(T_k - I)$ *for k=1,2,3,...,M is demiclosed at zero, then* $(U-I)$ *is also demiclosed at zero.*

In what follows, we adopt the following notations:

(i) I : The identity operator on H_1,

(ii) " $\to$ "and " $\rightharpoonup$ " The strong and weak covergence, respectively,

(iii) $\omega_\omega(x_n)$: The set of the cluster point of $\{x_n\}$ in the weak topology i.e., $\{$ there exists $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x\}$,

(iv) Γ : The solution set of Problem (2), i.e.,

$$\Gamma = \Big\{ \text{ Find } x^* \in \bigcap_{r=1}^{N} Fix(U_r) \text{ and } y^* \in \bigcap_{s=1}^{M} Fix(T_s) \text{ such that } Ax^* = By^* \Big\}. \tag{10}$$

3. Main Result

To approximate the solution of the SCFPEP (10), we make the following assumptions:

(C_1) H_1, H_2, H_3, are Hilbert spaces.

(C_2) $T_1, T_2, T_3, ..., T_N : H_1 \to H_1$ and $U_1, U_2, U_3, ..., U_M : H_2 \to H_2$ are firmly of quasi nonexpansive mappings with $\bigcap_{r=1}^{N} Fix(T_r) \neq \emptyset$ and $\bigcap_{r=1}^{M} Fix(U_s) \neq \emptyset$.

(C_3) $(T_r - I)$, for r=1,2,3,...,N and $(U_s - I)$, for r=1,2,3,...,M are demiclosed at zero.

(C_4) $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators with their adjoints A^* and B^*, respectively.

(C_5) For arbitrary $x_1 \in H_1$ and $y_1 \in H_2$, defined $\{(x_n, y_n)\}$ by:

$$\begin{cases} z_n = x_n - \lambda_n A^*(Ax_n - By_n), \\ w_n = (1-\gamma_n)z_n + \gamma_n \sum_{s=1}^{M} \delta_s U_{\beta_s}(z_n), \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n \sum_{s=1}^{M} \delta_s U_{\beta_s}(w_n), \\ \\ u_n = y_n + \lambda_n B^*(Ax_n - By_n), \\ r_n = (1-\gamma_n)u_n + \gamma_n \sum_{r=1}^{N} \lambda_r T_{\tau_r}(u_n), \\ y_{n+1} = (1-\alpha_n)u_n + \alpha_n \sum_{r=1}^{N} \lambda_r T_{\tau_r}(r_n), \forall n \geq 1. \end{cases} \tag{11}$$

where $U_{\beta_s} = (1-\beta_s)I + \beta_s U_s$ and $\beta_s \in (0,1)$, for s=1,2,3,...,M, $T_{\tau_r} = (1-\tau_r)I + \tau_r T_r$, and $\tau_r \in (0,1)$, for r=1,2,3,...,N, $\sum_{s=1}^{M} \delta_s = 1$ and $\sum_{r=1}^{N} \lambda_r = 1$, $0 < a < \gamma_n < 1$, $0 < b < \alpha_n < 1$ and $\lambda_n \in \left(0, \frac{2}{L_1+L_2}\right)$ where $L_1 = A^*A$ and $L_2 = B^*B$.

Theorem 3.1. *Suppose that conditions $(C_1) - (C_5)$ above are satisfied, also assume that the solution set $\Gamma \neq \emptyset$. Then $(x_n, y_n) \rightharpoonup (x^*, y^*) \in \Gamma$.*

Proof. Let $U = \sum_{s=1}^{M} \delta_s U_{\beta_s}$ and $T = \sum_{r=1}^{N} \lambda_r T_{\tau_r}$. By Lemma 2.2, we obtain that U and T are quasi nonexpansive mappings, $Fix(U) = \bigcap_{s=1}^{M} Fix(U_{\delta_s}) = \bigcap_{s=1}^{M} Fix(U_s)$ and $Fix(T) = \bigcap_{r=1}^{N} Fix(T_{\tau_r}) = \bigcap_{r=1}^{N} Fix(T_r)$, respectively. Let also $(x, y) \in \Gamma$, this implies that $x \in \bigcap_{r=1}^{N} Fix(U_r)$ and $y \in \bigcap_{s=1}^{M} Fix(T_s)$ such that $Ax = By$.

Thus, by (11), we deduce that

$$\begin{aligned}
\|w_n - x\|^2 &= \|(1-\gamma_n)(z_n - x) + \gamma_n(U(z_n) - x)\|^2 \\
&= (1-\gamma_n)\|z_n - x\|^2 + \gamma_n \|U(z_n) - x\|^2 - \gamma_n(1-\gamma_n)\|U(z_n) - z_n\|^2 \\
&\leq \|z_n - x\|^2 - \gamma_n(1-\gamma_n)\|U(z_n) - z_n\|^2 \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - x\|^2 &= \|x_n - \lambda_n A^*(Ax_n - By_n) - x\|^2 \\
&= \|x_n - x\|^2 - 2\lambda_n \langle Ax_n - Ax, Ax_n - By_n \rangle \\
&+ \lambda_n^2 L_1 \|Ax_n - By_n\|^2 . \qquad (13)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|x_{n+1} - x\|^2 &= \|(1-\alpha_n)(z_n - x) + \alpha_n(Uw_n - x)\|^2 \\
&= (1-\alpha_n)\|z_n - x\|^2 + \alpha_n\|Uw_n - x\|^2 - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 \\
&\leq (1-\alpha_n)\|z_n - x\|^2 + \alpha_n\|w_n - x\|^2 - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 \\
&\leq \|z_n - x\|^2 - \alpha_n\gamma_n(1-\gamma_n)\|U(z_n) - z_n\|^2 \\
&- \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 \text{ (by (12))} \\
&\leq \|x_n - x\|^2 - 2\lambda_n\langle Ax_n - Ax, Ax_n - By_n\rangle + \lambda_n^2 L_1\|Ax_n - By_n\|^2 \\
&- \alpha_n\gamma_n(1-\gamma_n)\|U(z_n) - z_n\|^2 \\
&- \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 \text{ (by (13))}. \qquad (14)
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
\|y_{n+1} - y\|^2 &\leq \|y_n - y\|^2 + 2\lambda_n\langle By_n - By, Ax_n - By_n\rangle + \lambda_n^2 L_2\|Ax_n - By_n\|^2 \\
&- \alpha_n\gamma_n(1-\gamma_n)\|T(u_n) - u_n\|^2 - \alpha_n(1-\alpha_n)\|Tr_n - u_n\|^2 . \qquad (15)
\end{aligned}$$

By (14), (15) and noticing that $Ax = By$, we deduce that

$$\begin{aligned}
\|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 &\leq \|x_n - x\|^2 + \|y_n - y\|^2 \\
&- \lambda_n\left(2 - \lambda_n^2(L_1 + L_2)\right)\|Ax_n - By_n\|^2 \\
&- \alpha_n\gamma_n(1-\gamma_n)\|U(z_n) - z_n\|^2 \\
&- \alpha_n\gamma_n(1-\gamma_n)\|T(u_n) - u_n\|^2 . \qquad (16)
\end{aligned}$$

Noticing that $\lambda_n\left(2 - \lambda_n^2(L_1 + L_2)\right) > 0$ and $\alpha_n\gamma_n(1-\gamma_n) > 0$, we deduce that

$$\|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 \leq \|x_n - x\|^2 + \|y_n - y\|^2 .$$

Thus, $\left\{\|x_n - x\|^2 + \|y_n - y\|^2\right\}$ is Fejer monotone, therefore, converges. This implies that $\{x_n\}$ and $\{y_n\}$ are bounded.

Let $(x, y) \in \Gamma$ such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, respectively. By equation (11), we have that $z_n \rightharpoonup x$ and $u_n \rightharpoonup y$.

Now, $z_n \rightharpoonup x$ and $\lim_{n\to\infty} \|Uz_n - z_n\| = 0$ together with the demiclosed of $(U - I)$ at zero, we deduce that $Ux = x$, this implies that $x \in Fix(U)$.

On the other hand, $u_n \rightharpoonup y$ and $\lim_{n\to\infty} \|Tu_n - u_n\| = 0$ together with the demiclosed of $(T - I)$ at zero, we deduce that $Ty = y$, this implies that $y \in Fix(T)$.

The fact that $z_n \rightharpoonup x$, $u_n \rightharpoonup y$, and together with the definitions A and B, we have

$$Az_n \rightharpoonup Ax \quad \text{and} \quad Bu_n \rightharpoonup By,$$

This implies that

$$Az_n - Bu_n \rightharpoonup Ax - By,$$

which turn to implies that

$$\|Ax - By\| \leq \liminf_{n\to\infty} \|Az_n - Bu_n\| = 0,$$

which further implies that $Ax = By$. Hence, we conclude that $(x, y) \in \Gamma$.

Thus, we have proved the following:

(i) for each $(x^*, y^*) \in \Gamma$, the $\lim_{n\to\infty}\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right)$ exists;

(ii) the weak cluster of the sequence (x_n, y_n) belongs to Γ.

Thus, by Lemma (2.1) we conclude that $(x_n, y_n) \rightharpoonup (x^*, y^*) \in \Gamma$. And the proof is complete. □

Corollary 3.2. *Suppose that conditions $(C_1) - (C_5)$ are satisfied, and let the sequence $\{(x_n, y_n)\}$ be defined by Algorithm (11). Assume that $\Gamma \neq \emptyset$ and let U and T be firmly quasi-nonexpansive mappings. Then $(x_n, y_n) \rightharpoonup (x^*, x^*) \in \Gamma$.*

4. Numerical Example

In this section, we illustrate the convergence result of Theorem 3.1 through the numerical example.

The following is an example of quasi-nonexpansive mapping.

Example 4.1. *Let* $H_1 = \Re$ *and* $H_2 = \Re$, $C := [0, \infty)$ *and* $Q := [0, \infty)$ *be subset of* H_1 *and* H_2*, respectively. Define* $T : C \to C$ *by* $Tx = \frac{x+2}{3}$ *for all* $x \in C$*, and* $U : Q \to Q$ *by*

$$Ux = \begin{cases} \frac{2x}{x+1}, \forall x \in [1, +\infty) \\ 0, \quad \forall x \in [0, 1). \end{cases} \tag{17}$$

Then, U *and* T *are quasi nonexpansive mappings.*

Proof. Trivially, $Fix(T) = 1$ and $Fix(U) = 1$.

Now,

$$\|Tx - 1\| = \left\| \frac{x+2}{3} - 1 \right\| \leq \|x - 1\|,$$

and

$$\|Ux - 1\| = \frac{1}{1+x} \|x - 1\| \leq \|x - 1\|.$$

Thus, U and T are quasi-nonexpansive mappings.

□

Example 4.2. *Let* $H_1 = \Re$ *and* $H_2 = \Re$, $C := [0, \infty)$ *and* $Q := [0, \infty)$ *be subset of* H_1 *and* H_2*, respectively. Define* $T : C \to C$ *by* $Tx = \frac{x+2}{3}$ *for all* $x \in C$*, and* $U : Q \to Q$ *by*

$$Ux = \begin{cases} \frac{2x}{x+1}, \forall x \in [1, +\infty) \\ 0, \quad \forall x \in [0, 1). \end{cases} \tag{18}$$

Let also $\lambda_n = 1$, $Ax = x$, $By = y$, $\beta_s = \frac{1}{3}$, $\tau_r = \frac{1}{5}$, $\alpha_n = \frac{1}{7}$ *and* $\gamma_n = \frac{1}{9}$*. The sequence* $\{(x_n, y_n)\}$ *defined by Algorithm 11 can be written as follows:*

$$\begin{cases} z_n = x_n - A^*(Ax_n - By_n), \\ w_n = \frac{8}{9}z_n + \frac{1}{9}\left(\frac{2z_n}{3} + \frac{2z_n}{3(z_n+1)}\right), \\ x_{n+1} = \frac{6}{7}z_n + \frac{1}{7}\left(\frac{2w_n}{3} + \frac{2w_n}{3(w_n+1)}\right), \\ \\ u_n = y_n + B^*(Ax_n - By_n), \\ r_n = \frac{8}{9}u_n + \frac{1}{9}\left(\frac{4u_n}{5} + \frac{u_n+2}{15}\right), \\ y_{n+1} = \frac{6}{7}u_n + \frac{1}{7}\left(\frac{4r_n}{5} + \frac{r_n+2}{15}\right), \forall n \geq 1. \end{cases} \tag{19}$$

Then $\{(x_n, y_n)\}$ converges to $(1,1) \in \Omega$. By Example 4.1, U and T are quasi-nonexpansive mappings with $Fix(U) = 1$ and $Fix(T) = 1$, respectively. Clearly, A, B are bounded linear on $\Re$, $A = A^* = 1$ and $B = B^* = 1$. Hence,

$$\Gamma = \{1 \in Fix(T) \text{ and } 1 \in Fix(U) \text{ such that } A(1) = B(1)\}.$$

Simplifying Algorithm (19), we have

$$\begin{cases} z_n = y_n, \\ w_n = \frac{8}{9}z_n + \frac{1}{9}\left(\frac{2z_n}{3} + \frac{2z_n}{3(z_n+1)}\right), \\ x_{n+1} = \frac{6}{7}z_n + \frac{1}{7}\left(\frac{2w_n}{3} + \frac{2w_n}{3(w_n+1)}\right), \\ \\ u_n = x_n, \\ r_n = \frac{8}{9}u_n + \frac{1}{9}\left(\frac{4u_n}{5} + \frac{u_n+2}{15}\right), \\ y_{n+1} = \frac{6}{7}u_n + \frac{1}{7}\left(\frac{4r_n}{5} + \frac{r_n+2}{15}\right), \forall n \geq 1. \end{cases} \tag{20}$$

We used Maple and obtained the numerical values of Algorithm 20 in the table 1 and table 2.

Table 1: Starting with initial values $x_0 = 5$ and $y_0 = 5$

n	x_n	y_n
0	5.000000000	5.000000000
1	4.916472663	4.760850019
2	4.834689530	4.537828465
3	4.754614179	4.329771078
.	.	.
.	.	.
.	.	.
148	1.176058095	1.007392532
149	1.172381679	1.007122340

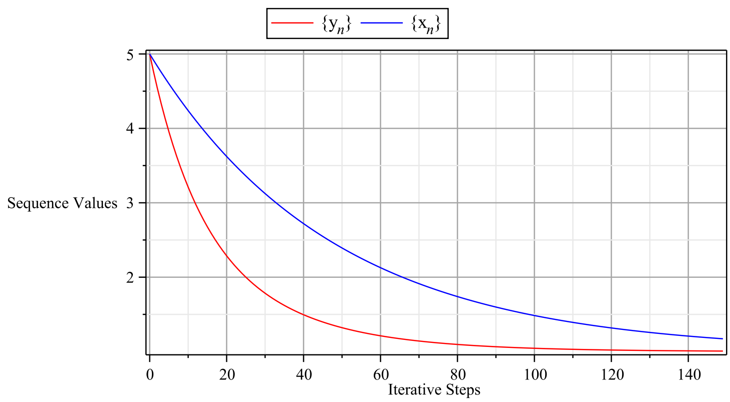

Figure 1: The convergence of $\{(x_n, y_n)\}$ with the initial value $x_0 = 5$ and $y_0 = 5$

Table 2: Starting with initial values $x_0 = -5$ and $y_0 = -5$

n	x_n	y_n
0	-5.000000000	-5.000000000
1	-4.460475401	-4.874708995
2	-3.953349994	-4.752034296
.	-3.474475616	-4.631921270
.	.	.
.	.	.
.	.	.
148	1.001346412	0 .7359128532
149	1.001297344	0.7414274772

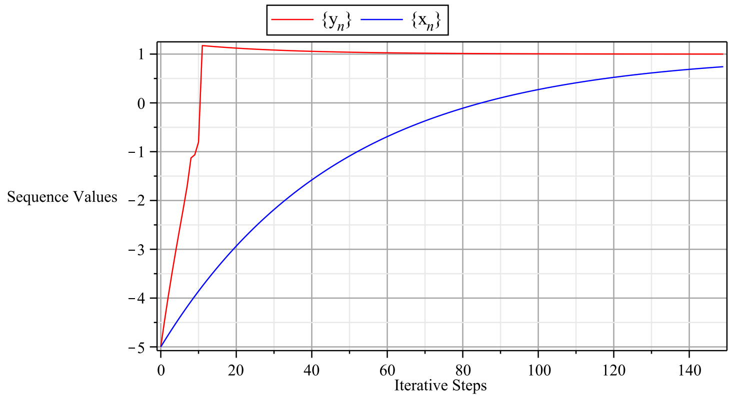

Figure 2: The convergence of $\{(x_n, y_n)\}$ with the initial value $x_0 = -5$ and $y_0 = -5$

5. Conclusion

In this paper, we studied the SCFPEP for the class of finite family of quasi-nonexpansive mappings in Hilbert spaces. Under some suitable assumptions imposed on the parameters and operators involved, we proved the weak convergence results for the proposed problem. Furthermore, we gave the numerical example that illustrates our theoretical result.

The SCFPEP is an interesting topic. It generalizes the split feasibility problem (SFP), split feasibility and fixed point problem (SFFPP) and split equality fixed point problem (SEFPP). All the results and conclusions that are true for the SCFPEP continue to hold for these problems (SFP, FPP, SFFPP, and SEFPP), and it shows the significance and the range of applicability of SCFPEP.

Remark 5.1. *Theorem 3.1 gives the weak convergence result for the solution of the SCFPEP for the class of finite family of quasi-nonexpansive mappings. We observed the strong convergence could follow easily by imposing the semi-compact conditions on some operators. However, this semi-compact condition is a strong assumption as only few mapping are semi-compact.*

This leads us to think of the following question:

Can the strong convergence of Theorem 3.1 be obtained without imposing the semi-compactness conditions? This will be our future research.

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